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Lagrangian moments and mass transport in Stokes waves

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The orbital motions in surface gravity waves are of interest for analysing wave records made by accelerometer buoys. In this paper we derive some exact expressions for the first, second and third cumulants of the vertical orbital displacements in a regular Stokes wave of finite amplitude in terms of previously known integral quantities of the wave: the kinetic and potential energies, the phase speed c and the mass-transport velocity U at the free surface. These results generalize a remarkably simple relation found previously between the Lagrangian-mean surface level and the product Uc.

Expansions are given in powers of the wave steepness parameter ak which show that the third Lagrangian cumulant is very small – of order $(ak)^6$, indicating a high degree of vertical symmetry in the orbit. This contrasts with the situation in random waves, where the third cumulant is of order $(ak)^4$ only. It is shown that the increased skewness in random waves is due mainly to an $O(ak)^2$ shift in the Lagrangian *mean* level of individual waves. Such shifts in mean level may be too gradual to be fully detected by some accelerometer buoys. In that case the apparent skewness will be reduced.

1. Introduction

With the widespread use of accelerometer buoys for measuring the surface elevation in ocean waves, there is some interest in understanding the Lagrangian properties of surface gravity waves. Recently a remarkable exact relation was found between the Lagrangian-mean elevation $\overline{\eta}_{\mathrm{L}}$, the Eulerian-mean elevation $\overline{\eta}_{\mathrm{E}}$ and the horizontal mass-transport velocity U at the free surface, namely

$$\overline{\eta}_{\mathrm{L}}-\overline{\eta}_{\mathrm{E}}=\frac{Uc}{2g}, \tag{1.1}$$

where c is the phase speed and g denotes gravity. This was proved in Longuet-Higgins (1986), and is valid for uniform Stokes waves of any finite steepness ak.

The purpose of the present note is, first, to point out that the second and third moments of η_{L} can likewise be related exactly to the mass-transport velocity and to other known integral quantities of the motion, such as the kinetic and potential energies. This is done in §3, below.

A second purpose is to evaluate the Lagrangian cumulants numerically and derive their expansions in powers of ak (§§4 and 5). In particular it is found that the third cumulant of the Lagrangian elevation η_{L} is very small indeed – of order $(ak)^6$, implying that the particle orbit is highly symmetric in the vertical direction.

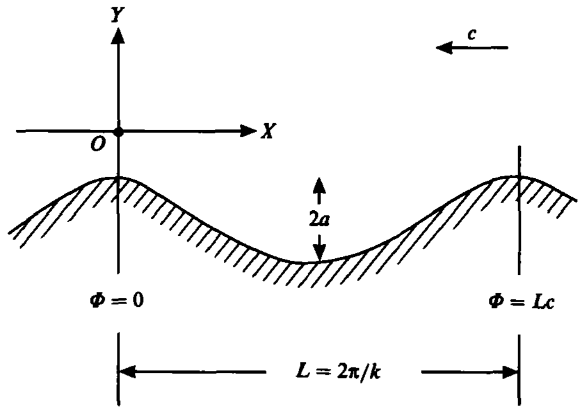

FIGURE 1. Coordinates and notation for a Stokes wave in deep water.

The situation in random waves is discussed in §6. Paradoxically, it was shown by Srokosz & Longuet-Higgins (1986) that in this situation the third Lagrangian cumulant is of lower order $-(ak)^4$ not $(ak)^6$. Our analysis confirms the physical explanation given in that paper. For, the local mean level of the Lagrangian orbit is displaced by $O(ak)^2$ from the overall mean level $\tilde{\eta}_{\mathrm{L}}$. It follows that the third moment about $\tilde{\eta}_{\mathrm{L}}$ is $O(ak)^4$ only.

Practical implications of the results are discussed in §7.

2. Uniform Stokes waves

Consider a regular train of irrotational gravity waves in an inviscid, incompressible fluid of uniform density, and infinite depth, as in figure 1. Take rectangular axes travelling with the phase-speed c, the origin O being above a wave crest, at a level such that in Bernoulli's equation

$$\frac{p}{\rho}+\tfrac{1}{2}q^2+gy = \text{constant} \tag{2.1}$$

the constant on the right vanishes. Here p denotes the pressure, ρ the density and q the particle speed. At the free surface p vanishes and so if η denotes the surface elevation we have

$$\tfrac{1}{2}q^2+g\eta = 0. \tag{2.2}$$

By the argument given in Lamb (1932, p. 420), the Eulerian mean level is given by

$$\bar{\eta}_{\mathrm{E}} = -\frac{c^2}{2g}. \tag{2.3}$$

The second Eulerian moment about the mean is by definition

$$\overline{(\eta_{\mathrm{E}}-\bar{\eta}_{\mathrm{E}})^2} = \frac{2V}{g}, \tag{2.4}$$

where V is the potential energy density. Hence

$$\overline{\eta_{\mathrm{E}}^2} = \frac{2V}{g}+\frac{c^4}{4g^2}. \tag{2.5}$$

For the third and higher moments there appear to be no simple closed expressions.

Consider now the Lagrangian moments. If t denotes the time following a particle we have in general

$$\mathrm{d}t = q^{-2}\,\mathrm{d}\Phi \tag{2.6}$$

where Φ is the velocity potential in the steady motion. Thus the Lagrangian wave period is

$$T_{\mathrm{L}} = \int \mathrm{d}t = \int q^{-2}\,\mathrm{d}\Phi, \tag{2.7}$$

the integral being taken over a complete orbit, or a wavelength of the steady motion. The rth Lagrangian moment of η is, by definition,

$$\overline{\eta_{\mathrm{L}}^{r}} = \frac{1}{T_{\mathrm{L}}}\int \eta^{r}\,\mathrm{d}t = \frac{1}{T_{\mathrm{L}}}\int \frac{\eta^{r}}{q^{2}}\,\mathrm{d}\Phi \tag{2.8}$$

by (2.6). Substituting for q^2 from (2.2) we find

$$\overline{\eta_{\mathrm{L}}^{r}} = -\frac{1}{2gT_{\mathrm{L}}}\int \eta^{r-1}\,\mathrm{d}\Phi. \tag{2.9}$$

In the particular case $r = 1$ we have, since

$$\int \mathrm{d}\Phi = cL = c^{2}T_{\mathrm{E}} \tag{2.10}$$

(where T_{E} is the Eulerian wave period), that

$$\overline{\eta}_{\mathrm{L}} = -\frac{c^{2}}{2g}\frac{T_{\mathrm{E}}}{T_{\mathrm{L}}}. \tag{2.11}$$

But if U denotes the mass-transport velocity at the surface, then

$$\frac{T_{\mathrm{E}}}{T_{\mathrm{L}}} = \frac{L/c}{L/(c-U)} = 1-\frac{U}{c} \tag{2.12}$$

and so subtracting (2.11) from (2.3) we get the result (1.1). We shall now consider higher values of r.

3. Higher moments

It is convenient to set $g = 1$, $k = 1$ and to introduce the Fourier expansions

$$\left.\begin{aligned} \eta &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\Phi}{c}\right), \\ X &= \frac{\Phi}{c} + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\Phi}{c}\right), \end{aligned}\right\} \tag{3.1}$$

where Φ is the velocity potential in the steady motion and the a_n are the well-known Stokes coefficients. As in Longuet-Higgins (1985) we may also define the quantities

$$\left.\begin{aligned} J &= \tfrac{1}{2}(a_1^2 + a_2^2 + \dots), \\ K &= \tfrac{1}{2}(a_1 b_1 + a_2 b_2 + \dots), \end{aligned}\right\} \tag{3.2}$$

where $b_n = na_n$, $n \geqslant 1$. It is known that

$$K + \tfrac{1}{2}a_0 = -\tfrac{1}{2}c^{2} = \overline{\eta}_{\mathrm{E}} \tag{3.3}$$

and also that
$$\left.\begin{aligned} 2T &= c^2K, \\ 2V &= J+2c^2K+K^2, \end{aligned}\right\} \tag{3.4}$$

T being the kinetic energy density (see Longuet-Higgins 1984, 1985).

Consider the case $r = 2$. From (2.9) we have

$$\overline{\eta_{\mathrm{L}}^2} = -\frac{1}{2T_{\mathrm{L}}}\int \eta\,\mathrm{d}\Phi \tag{3.5}$$

and so by (3.1) and (2.10)

$$\overline{\eta_{\mathrm{L}}^2} = -\tfrac{1}{2}c^2\,\frac{T_{\mathrm{E}}}{T_{\mathrm{L}}}\,\tfrac{1}{2}a_0. \tag{3.6}$$

Using (3.3) we may write this as

$$\overline{\eta_{\mathrm{L}}^2} = \tfrac{1}{2}c^2\,\frac{T_{\mathrm{E}}}{T_{\mathrm{L}}}\,(\tfrac{1}{2}c^2+K) \tag{3.7}$$

or from (3.4)
$$\overline{\eta_{\mathrm{L}}^2} = \tfrac{1}{2}c^2$$

$$\overline{\eta_{\mathrm{L}}^2} = \tfrac{1}{2}c^2\,\frac{T_{\mathrm{E}}}{T_{\mathrm{L}}}\left(\tfrac{1}{2}c^2+\frac{2T}{c^2}\right). \tag{3.8}$$

(Alternatively in (3.5) we may note that $\Phi = \phi - cx$, hence $\mathrm{d}\Phi = \mathrm{d}\phi - c\,\mathrm{d}x$, and then use the expression $\int \eta\,\mathrm{d}\phi$ for the kinetic energy T.)

Similarly in the case $r = 3$ we have from (2.9)

$$\overline{\eta_{\mathrm{L}}^3} = -\frac{1}{2T_{\mathrm{L}}}\int \eta^2\,\mathrm{d}\Phi. \tag{3.9}$$

Substitution from the series (3.1) then gives

$$\overline{\eta_{\mathrm{L}}^3} = -\tfrac{1}{2}c^2\,\frac{T_{\mathrm{E}}}{T_{\mathrm{L}}}\,(\tfrac{1}{4}a_0^2+J). \tag{3.10}$$

Using (3.3) and (3.4) we obtain

$$\overline{\eta_{\mathrm{L}}^3} = -\tfrac{1}{2}c^2\,\frac{T_{\mathrm{E}}}{T_{\mathrm{L}}}\,(6V-2T+\tfrac{1}{4}c^4). \tag{3.11}$$

Since $T_{\mathrm{E}}/T_{\mathrm{L}}$ is already given in terms of U/c by (2.12), (3.8) and (3.11) express both $\overline{\eta_{\mathrm{L}}^2}$ and $\overline{\eta_{\mathrm{L}}^3}$ in terms of c^2, T, V and U/c.

It may be noted that if we complete the definitions (3.2) by writing

$$N = \tfrac{1}{2}(b_1^2+b_2^2+\dots), \tag{3.12}$$

then since
$$q^{-2} = X_{\phi\phi}^2+\eta_{\phi\phi}^2 \tag{3.13}$$

we have from (3.1), on substitution in (2.7),

$$T_{\mathrm{L}} = \frac{L}{c}\,(1+b_1^2+b_2^2+\dots), \tag{3.14}$$

cf. Ursell (1953). Hence

$$\frac{T_{\mathrm{L}}}{T_{\mathrm{E}}} = 1+2N \tag{3.15}$$

and
$$\frac{U}{c} = 1-\frac{T_{\mathrm{E}}}{T_{\mathrm{L}}} = \frac{2N}{1+2N}. \tag{3.16}$$

ak	c^2-1	J	K	N	T	V	U/c	κ_{L2}	κ_{L3}
0	0	0	0	0	0	0	0	0	0
.1	.01005	.00490	.00495	.00505	.00250	.00250	.01000	.00500	.00000
.2	.04081	.01837	.01916	.02088	.00997	.00977	.04009	.01999	.00001
.3	.09415	.03655	.04040	.05028	.02210	.02110	.09137	.04493	.00010
.35	.13002	.04512	.05210	.07309	.02944	.02760	.12691	.06108	.00031
.4	.17121	.05100	.06236	.01510	.03652	.03349	.17369	.07939	.00071
.42	.18719	.05162	.06487	.12491	.03851	.03498	.19988	.08716	.00088
.44316	.19308	.05012	.064190	.18830	.03829	.03457	.27357	.09854	.00143

TABLE 1. Parameters of uniform gravity waves in deep water

Altogether, the three parameters J, K and N can be expressed in terms of physical quantities by

$$\left.\begin{aligned} J &= 6V-4T-\frac{4T^2}{c^4}, \\ K &= \frac{2T}{c^2}, \\ N &= \frac{1}{2}(c/U-1). \end{aligned}\right\} \tag{3.17}$$

4. The Lagrangian cumulants

For further progress it is convenient to set

$$\left.\begin{aligned} A &= -\tfrac{1}{2}a_0, \\ B &= 1-\frac{U}{c}, \\ C &= \tfrac{1}{2}c^2, \end{aligned}\right\} \tag{4.1}$$

so that we have

$$\left.\begin{aligned} \overline{\eta}_{\mathrm{L}} &= -BC, \\ \overline{\eta^2_{\mathrm{L}}} &= BCA, \\ \overline{\eta^3_{\mathrm{L}}} &= -BC(A^2+J). \end{aligned}\right\} \tag{4.2}$$

It follows immediately that the first three Lagrangian cumulants are given by

$$\left.\begin{aligned} \kappa_{\mathrm{L1}} &= 0, \\ \kappa_{\mathrm{L2}} &= \overline{\eta^2_{\mathrm{L}}}-\overline{\eta}^2_{\mathrm{L}} = BC(A-BC), \\ \kappa_{\mathrm{L3}} &= \overline{\eta^3_{\mathrm{L}}}-3\overline{\eta^2_{\mathrm{L}}}\,\overline{\eta}_{\mathrm{L}}+2\overline{\eta}^3_{\mathrm{L}} \\ &= -BC[(A-BC)(A-2BC)+J]. \end{aligned}\right\} \tag{4.3}$$

Numerical values of c^2, J, K, N, T, V and U/c are given in table 1, for selected values of the wave amplitude

$$a = a_1+a_3+a_5+\dots. \tag{4.4}$$

For $a = 0.1, 0.2, 0.3, 0.35, 0.4$ and 0.42 we used the method of computation described in Longuet-Higgins (1985). The values for the limiting wave $a = 0.443\dots$ are derived

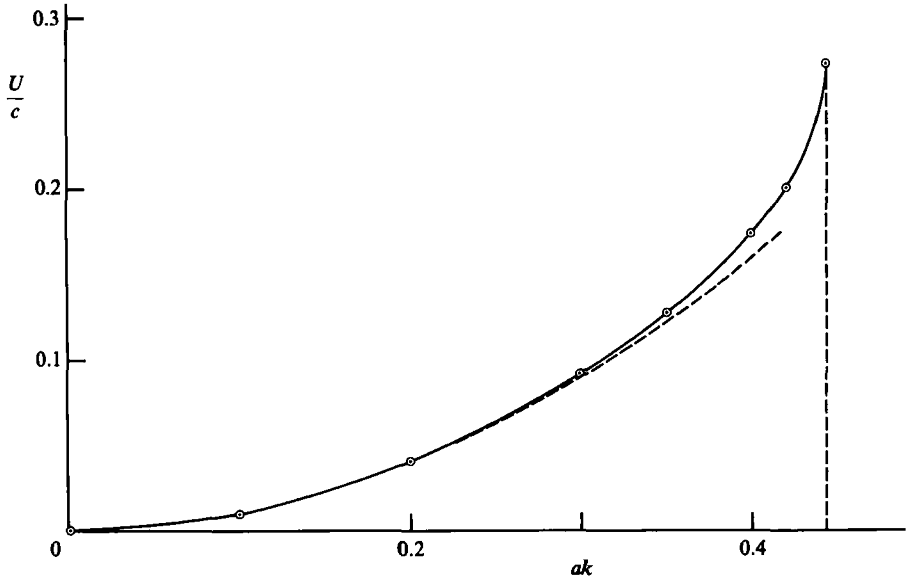

FIGURE 2. The mass-transport velocity U at the free surface of an irrotational gravity wave, as a function of the wave steepness parameter ak. ——, Exact theory, equation (3.16); – – – –, small-amplitude approximation, $U/c = (ak)^2$.

from the values of c^2, T and V given by Williams (1981), and the value of U/c found by Longuet-Higgins (1979).

The ratio U/c is shown graphically by the full curve in figure 2. Near the limiting steepness $a_{max} = 0.443 \ldots$ the theory of the almost-highest wave (Longuet-Higgins 1979, 1986) indicates that $(T_L)_{max} - T_L$ varies like ϵT_E where ϵ is a small parameter proportional to $(a_{max} - a)^{\frac{1}{2}}$. It follows that the tangent to the curve of U/c at $a = a_{max}$ must be vertical.

The behaviour of the other parameters c^2, T, V, etc. in the neighbourhood of the limiting steepness is known to be oscillatory (Longuet-Higgins & Fox 1978) and may be determined from the asymptotic formulae given in that paper.

The cumulants κ_{L2} and κ_{L3} are tabulated also on the right of table 1. From the final column it will be seen how very small is the third cumulant κ_{L3}. Even for the steepest wave, it is of order 10^{-3} only. In this extreme case, the coefficient of skewness

$$\lambda_{L3} = \frac{\kappa_{L3}}{\kappa_{L2}^{\frac{3}{2}}} \tag{4.5}$$

is still less than five percent.

5. Series expansions

For waves of low or moderate steepness ak it is often convenient to have approximate expressions for the wave parameters in powers of ak. The simplest way to obtain such expansions is through the system of equations

$$\left.\begin{aligned} a_1 + a_0 b_1 + a_1 b_2 + a_2 b_3 + \ldots &= 0, \\ a_2 + a_1 b_1 + a_0 b_2 + a_1 b_3 + \ldots &= 0, \\ a_3 + a_2 b_1 + a_1 b_2 + a_0 b_3 + \ldots &= 0, \\ \ldots \quad & \end{aligned}\right\} \tag{5.1}$$

found by Longuet-Higgins (1978). Using the expansion procedure given there, and setting $k = 1$, we easily obtain

$$\left.\begin{aligned} a_0 &= -1-2a^2+\tfrac{1}{2}a^4+\dots, \\ a_1 &= a-\tfrac{3}{2}a^3-\tfrac{1}{24}a^5+\dots, \\ a_2 &= a^2-\tfrac{5}{2}a^4+\dots, \\ a_3 &= \tfrac{3}{2}a^3-\tfrac{31}{6}a^5+\dots, \\ a_4 &= \tfrac{8}{3}a^4+\dots, \\ a_5 &= \tfrac{125}{24}a^5+\dots, \end{aligned}\right\} \tag{5.2}$$

as far as terms in a^5. From the definitions of §3 we than have

$$\left.\begin{aligned} J &= \tfrac{1}{2}a^2-a^4-\tfrac{7}{24}a^6+\dots, \\ K &= \tfrac{1}{2}a^2-\tfrac{1}{2}a^4-\tfrac{13}{24}a^6+\dots, \\ N &= \tfrac{1}{2}a^2+\tfrac{1}{2}a^4+\tfrac{29}{24}a^6+\dots, \end{aligned}\right\} \tag{5.3}$$

from which it follows that

$$\left.\begin{aligned} c^2-1 &= a^2 +\tfrac{1}{2}a^4+\tfrac{1}{4}a^6+\dots, \\ T &= \tfrac{1}{4}a^2 \qquad -\tfrac{19}{48}a^6+\dots, \\ V &= \tfrac{1}{4}a^2-\tfrac{1}{8}a^4-\tfrac{19}{48}a^6+\dots, \\ U/c &= a^2 \qquad +\tfrac{17}{12}a^6+\dots. \end{aligned}\right\} \tag{5.4}$$

The first three expansions agree with previously known results (Longuet-Higgins 1975, §6). The fourth of equations (5.4) shows that for moderate values of a the approximation $U/c = a^2$ is very close indeed (see figure 2).

Lastly from (4.2) and (4.3) we have for the mean

$$\overline{\eta}_{\mathrm{L}} = -\tfrac{1}{2}+\tfrac{1}{4}a^4+\tfrac{5}{6}a^6+\dots, \tag{5.5}$$

and for the second and third cumulants

$$\left.\begin{aligned} \kappa_{\mathrm{L}2} &= \tfrac{1}{2}a^2-\tfrac{1}{24}a^6+\dots, \\ \kappa_{\mathrm{L}3} &= \qquad \tfrac{1}{8}a^6+\dots. \end{aligned}\right\} \tag{5.6}$$

These show that both $\overline{\eta}_{\mathrm{L}}$ and $\kappa_{\mathrm{L}2}$ are given accurately by their lowest-order terms. The most notable feature, however is the smallness of the third cumulant $\kappa_{\mathrm{L}3}$, which is of order a^6 only. This indicates a remarkable degree of symmetry, with a coefficient of skewness $\lambda_{\mathrm{L}3} \doteqdot a^3/8^{\frac{1}{2}}$ only.

Equations (5.5) and (5.6) may be compared with the Eulerian quantities

$$\overline{\eta}_{\mathrm{E}} = -\tfrac{1}{2}-\tfrac{1}{2}a^2-\tfrac{1}{4}a^4-\tfrac{1}{8}a^6+\dots \tag{5.7}$$

and

$$\left.\begin{aligned} \kappa_{\mathrm{E}2} &= \tfrac{1}{2}a^2-\tfrac{1}{4}a^4-\tfrac{19}{26}a^6+\dots, \\ \kappa_{\mathrm{E}3} &= \tfrac{3}{4}a^4+\tfrac{43}{16}a^6+\dots, \end{aligned}\right\} \tag{5.8}$$

which differ significantly from $\kappa_{\mathrm{L}2}$ and $\kappa_{\mathrm{L}3}$. The third moment, in particular is an order a^4, not a^6, so that the coefficient of skewness $\lambda_{\mathrm{E}3} = 3a/2^{\frac{1}{2}}$, relatively large compared to the Lagrangian skewness.

6. Random waves

The above results apply strictly to waves of uniform amplitude a. In random waves the conclusions are somewhat different. If we assume, as in Srokosz & Longuet-Higgins (1986), that the dominant waves have a fairly narrow spectrum, with a Rayleigh distribution of wave amplitudes a, then in deep water it can be shown (see Longuet-Higgins & Stewart 1964) that it is the Eulerian mean level $\bar{\eta}_E$ that is constant to order a^2, not the Lagrangian mean $\bar{\eta}_L$. These two levels differ, as we have seen, by an amount

$$\Delta = \bar{\eta}_L - \bar{\eta}_E = \tfrac{1}{2}a^2 \tag{6.1}$$

to lowest order. Now the local contributions of a wave of amplitude a to the Lagrangian moments about the Eulerian mean level $\bar{\eta}_E$ are clearly

$$\left.\begin{aligned} m_{L1} &= \overline{(\eta_L - \bar{\eta}_E)} = \overline{(\eta_L - \bar{\eta}_L + \Delta)} = \Delta \doteqdot \tfrac{1}{2}a^2, \\ m_{L2} &= \overline{(\eta_L - \bar{\eta}_E)^2} = \kappa_{L2} + \Delta^2 \doteqdot \tfrac{1}{2}a^2, \\ m_{L3} &= \overline{(\eta_L - \bar{\eta}_E)^3} = \kappa_{L3} + 3\kappa_{L2}\Delta + \Delta^3 \doteqdot \tfrac{3}{4}a^4, \end{aligned}\right\} \tag{6.2}$$

the largest contribution to m_{L3} coming from $3\kappa_{L2}\,\Delta$, since κ_{L3} is of order a^6 only.

To find the total contribution to the moments $\tilde{m}_{Lr}$ for *non-uniform* waves we average the expressions (6.2) with respect to the Rayleigh density

$$p(a) = \frac{2a}{\alpha^2}\, \mathrm{e}^{-a^2/\alpha^2}, \tag{6.3}$$

in which α denotes the root-mean-square wave amplitude:

$$\alpha = (\tilde{a}^2)^{\frac{1}{2}}. \tag{6.4}$$

(Here we use a tilde to denote the average over many wave groups.) Thus we obtain

$$\tilde{m}_{L1} = \tfrac{1}{2}\alpha^2, \quad \tilde{m}_{L2} = \tfrac{1}{2}\alpha^2, \quad \tilde{m}_{L3} = \tfrac{3}{2}\alpha^4. \tag{6.5}$$

Finally, to find the corresponding cumulants, which to third order are simply moments about the mean $\tilde{m}_L^{(1)}$, we have

$$\left.\begin{aligned} \tilde{\kappa}_{L1} &= 0 \quad \text{(by definition)}, \\ \tilde{\kappa}_{L2} &= \tilde{m}_{L2} - \tilde{m}_{L1}^2 = \tfrac{1}{4}\alpha^2, \\ \tilde{\kappa}_{L3} &= \tilde{m}_{L3} - 3\tilde{m}_{L2}\,\tilde{m}_{L1} + 2\tilde{m}_{L1}^3 \\ &= \tfrac{3}{4}\alpha^4. \end{aligned}\right\} \tag{6.6}$$

Hence the Lagrangian skewness, in random waves, is of order a^4, not a^6.

The above analysis confirms the physical interpretation given by Srokosz & Longuet-Higgins (1986). For, the Lagrangian orbits are individually highly symmetric, in waves of any given amplitude, the third moments being of order a^6. However, the larger waves make a contribution whose mean is shifted positively by order a^2 from the overall mean. Thus the 'tails' of the overall distribution are shifted relatively to the right, producing a positive third cumulant and a positive coefficient of skewness.

7. Conclusions and applications

We have shown that for regular Stokes waves in deep water the Lagrangian moments and cumulants are given exactly by the formulae (4.2) and (4.3), in which A, B, C and J are previously known integral quantities of the motion. These lead directly to the expansions (5.6) which demonstrate immediately the smallness of κ_{L3} and consequently the high degree of symmetry in the vertical Lagrangian displacement.

However, in a random sea, despite the symmetry in the individual orbits, the Lagrangian skewness is much greater; theoretically it is equal to the Eulerian skewness, to lowest order. We re-emphasize, however, that the result is valid only if the integrating circuit used with the accelerometer has a time constant that is long compared to the wave groups. Otherwise the measured skewness will be reduced by an amount that is almost proportional to the frequency response at the mean group length of the waves.

As pointed out elsewhere (Longuet-Higgins 1986; Srokosz & Longuet-Higgins 1986) both the mean level $\bar{\eta}_L$ and the skewness parameter λ_{E3} may be of importance in applications of remote sensing of the oceans from aircraft and satellites. In this connection we note that the precise formula (1.1) may be quite useful since it can readily be generalized to random seas, to order a^2, and the surface mass-transport velocity U is easily measured from aerial observations, at least in swell.

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